

QCD in terms of gauge-invariant dynamical variables

Hans-Peter Pavel*

Institut für Kernphysik, TU Darmstadt, D-64289 Darmstadt, Germany

Bogoliubov Laboratory of Theoretical Physics, JINR Dubna, Russia

E-mail: hans-peter.pavel@physik.tu-darmstadt.de

For a complete description of the physical properties of low-energy QCD, it might be advantageous to first reformulate QCD in terms of gauge-invariant dynamical variables, before applying any approximation schemes. Using a canonical transformation of the dynamical variables, which Abelianises the non-Abelian Gauss-law constraints to be implemented, such a reformulation can be achieved for QCD. The exact implementation of the Gauss laws reduces the colored spin-1 gluons and spin-1/2 quarks to unconstrained colorless spin-0, spin-1, spin-2 and spin-3 glueball fields and colorless Rarita-Schwinger fields respectively. The obtained physical Hamiltonian can then be rewritten into a form, which separates the rotational from the scalar degrees of freedom, and admits a systematic strong-coupling expansion in powers of $\lambda = g^{-2/3}$, equivalent to an expansion in the number of spatial derivatives. The leading-order term in this expansion corresponds to non-interacting hybrid-glueballs, whose low-lying masses can be calculated with high accuracy by solving the Schrödinger-equation of the Dirac-Yang-Mills quantum mechanics of spatially constant physical fields (at the moment only for the 2-color case). Due to the presence of classical zero-energy valleys of the chromomagnetic potential for two arbitrarily large classical glueball fields (the unconstrained analogs of the well-known constant Abelian fields), practically all glueball excitation energy is expected to go into the increase of the strengths of these two fields. Higher-order terms in λ lead to interactions between the hybrid-glueballs and can be taken into account systematically using perturbation theory in λ .

Xth Quark Confinement and the Hadron Spectrum

8-12 October 2012

TUM Campus Garching, Munich, Germany

*Speaker.

1. Introduction

The QCD action

$$\mathcal{S}[A, \psi, \bar{\psi}] = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \right] \quad (1.1)$$

is invariant under the $SU(3)$ gauge transformations $U[\omega(x)] \equiv \exp(i\omega_a \tau_a/2)$

$$\psi^\omega(x) = U[\omega(x)] \psi(x), \quad A_{a\mu}^\omega(x) \tau_a/2 = U[\omega(x)] \left(A_{a\mu}(x) \tau_a/2 + \frac{i}{g} \partial_\mu \right) U^{-1}[\omega(x)]. \quad (1.2)$$

Introducing the chromoelectric $E_i^a \equiv F_{i0}^a$ and chromomagnetic $B_i^a \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}^a$ and noting that the momenta conjugate to the spatial A_{ai} are $\Pi_{ai} = -E_{ai}$, one obtains the canonical Hamiltonian

$$H_C = \int d^3x \left[\frac{1}{2} E_{ai}^2 + \frac{1}{2} B_{ai}^2(A) - g A_{ai} j_{ia}(\psi) + \bar{\psi} (\gamma_i \partial_i + m) \psi - g A_{a0} (D_i(A)_{ab} E_{bi} - \rho_a(\psi)) \right], \quad (1.3)$$

with the covariant derivative $D_i(A)_{ab} \equiv \delta_{ab} \partial_i - g f_{abc} A_{ci}$ in the adjoint representation.

Exploiting the *time dependence of the gauge transformations* (1.2) to put (see e.g. [1])

$$A_{a0} = 0, \quad a = 1, \dots, 8 \quad (\text{Weyl gauge}), \quad (1.4)$$

and quantising the dynamical variables A_{ai} , $-E_{ai}$, $\psi_{\alpha r}$ and $\psi_{\alpha r}^*$ in the Schrödinger functional approach by imposing equal-time (anti-) commutation relations (CR), e.g. $-E_{ai} = -i\partial/\partial A_{ai}$, the physical states Φ have to satisfy both the Schrödinger equation and the Gauss laws

$$H\Phi = \int d^3x \left[\frac{1}{2} E_{ai}^2 + \frac{1}{2} B_{ai}^2[A] - A_{ai} j_{ia}(\psi) + \bar{\psi} (\gamma_i \partial_i + m) \psi \right] \Phi = E\Phi, \quad (1.5)$$

$$G_a(x)\Phi = [D_i(A)_{ab} E_{bi} - \rho_a(\psi)] \Phi = 0, \quad a = 1, \dots, 8. \quad (1.6)$$

The Gauss law operators G_a are the generators of the residual *time independent gauge transformations* in (1.2), satisfying $[G_a(x), H] = 0$ and $[G_a(x), G_b(y)] = i f_{abc} G_c(x) \delta(x-y)$.

Furthermore, H commutes with the angular momentum operators

$$J_i = \int d^3x [-\epsilon_{ijk} A_{aj} E_{ak} + \Sigma_i(\psi) + \text{orbital parts}], \quad i = 1, 2, 3. \quad (1.7)$$

The matrix element of an operator O is given in the *Cartesian* form

$$\langle \Phi' | O | \Phi \rangle \propto \int dA d\bar{\psi} d\psi \Phi'^*(A, \bar{\psi}, \psi) O \Phi(A, \bar{\psi}, \psi). \quad (1.8)$$

The spectrum of Equ.(1.5)-(1.6) for the case of Yang-Mills quantum mechanics of spatially constant gluon fields, has been found in [2] for $SU(2)$ and in [3] for $SU(3)$, in the context of a weak coupling expansion in $g^{2/3}$, using the variational approach with gauge-invariant wave-functionals automatically satisfying (1.6). The corresponding unconstrained approach, a description in terms of gauge-invariant dynamical variables via an exact implementation of the Gauss laws, has been considered by many authors (o.a. [1],[4]-[10], and references therein) to obtain a non-perturbative description of QCD at low energy, as an alternative to lattice QCD.

I shall first discuss in Section 2 the unphysical, but technically much simpler case of 2-colors, and then show in Section 3 how the results can be generalised to $SU(3)$.

2. Unconstrained Hamiltonian formulation of 2-color QCD

2.1 Canonical transformation to adapted coordinates

Point transformation from the A_{ai}, ψ_α to a new set of adapted coordinates, the 3 angles q_j of an orthogonal matrix $O(q)$, the 6 elements of a pos. definite symmetric 3×3 matrix S , and new ψ'_β

$$A_{ai}(q, S) = O_{ak}(q) S_{ki} - \frac{1}{2g} \varepsilon_{abc} (O(q) \partial_i O^T(q))_{bc}, \quad \psi_\alpha(q, \psi') = U_{\alpha\beta}(q) \psi'_\beta, \quad (2.1)$$

where the orthogonal $O(q)$ and the unitary $U(q)$ are related via $O_{ab}(q) = \frac{1}{2} \text{Tr}(U^{-1}(q) \tau_a U(q) \tau_b)$. Equ. (2.1) is the generalisation of the (unique) polar decomposition of A and corresponds to

$$\chi_i(A) = \varepsilon_{ijk} A_{jk} = 0 \quad (\text{"symmetric gauge"}). \quad (2.2)$$

Preserving the CR, we obtain the old canonical momenta in terms of the new variables

$$-E_{ai}(q, S, p, P) = O_{ak}(q) \left[P_{ki} + \varepsilon_{kil} {}^*D_{ls}^{-1}(S) \left(\Omega_{sj}^{-1}(q) p_j + \rho_s(\psi') + D_n(S)_{sm} P_{mn} \right) \right]. \quad (2.3)$$

In terms of the new canonical variables the Gauss law constraints are Abelianised,

$$G_a \Phi \equiv O_{ak}(q) \Omega_{ki}^{-1}(q) p_i \Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i} \Phi = 0 \quad (\text{Abelianisation}), \quad (2.4)$$

and the angular momenta become

$$J_i = \int d^3x \left[-2\varepsilon_{ijk} S_{mj} P_{mk} + \Sigma_i(\psi') + \rho_i(\psi') + \text{orbital parts} \right]. \quad (2.5)$$

Equ.(2.4) identifies the q_i with the gauge angles and S and ψ' as the physical fields. Furthermore, from Equ.(2.5) follows that the S are *colorless* spin-0 and spin-2 glueball fields, and the ψ' *colorless* reduced quark fields of spin-0 and spin-1. Hence the gauge reduction corresponds to the conversion "*color* \rightarrow *spin*". The obtained unusual spin-statistics relation is specific to SU(2).

2.2 Physical quantum Hamiltonian

According to the general scheme [1], the *correctly ordered physical quantum Hamiltonian* in terms of the physical variables $S_{ik}(\mathbf{x})$ and the canonically conjugate $P_{ik}(\mathbf{x}) \equiv -i\delta/\delta S_{ik}(\mathbf{x})$ reads [8]

$$\begin{aligned} H(S, P) = & \frac{1}{2} \mathcal{J}^{-1} \int d^3\mathbf{x} P_{ai} \mathcal{J} P_{ai} + \frac{1}{2} \int d^3\mathbf{x} \left[B_{ai}^2(S) - S_{ai} j_{ia}(\psi') + \bar{\Psi}' (\gamma_i \partial_i + m) \Psi' \right] \\ & - \mathcal{J}^{-1} \int d^3\mathbf{x} \int d^3\mathbf{y} \left\{ \left(D_i(S)_{ma} P_{im} + \rho_a(\psi') \right) (\mathbf{x}) \mathcal{J} \right. \\ & \left. \langle \mathbf{x} a | {}^*D^{-2}(S) | \mathbf{y} b \rangle \left(D_j(S)_{bn} P_{nj} + \rho_b(\psi') \right) (\mathbf{y}) \right\}, \quad (2.6) \end{aligned}$$

with the Faddeev-Popov (FP) operator

$${}^*D_{kl}(S) \equiv \varepsilon_{kmi} D_i(S)_{ml} = \varepsilon_{kli} \partial_i - g(S_{kl} - \delta_{kl} \text{tr} S), \quad (2.7)$$

and the Jacobian $\mathcal{J} \equiv \det |{}^*D|$. The matrix element of a physical operator O is given by

$$\langle \Psi' | O | \Psi \rangle \propto \int_{S \text{ pos. def.}} \int_{\bar{\Psi}', \Psi'} \prod_{\mathbf{x}} \left[dS(\mathbf{x}) d\bar{\Psi}'(\mathbf{x}) d\Psi'(\mathbf{x}) \right] \mathcal{J} \Psi'^*[S, \bar{\Psi}', \Psi'] O \Psi[S, \bar{\Psi}', \Psi']. \quad (2.8)$$

The inverse of the FP operator and hence the physical Hamiltonian can be expanded in the number of spatial derivatives, equivalent to a strong coupling expansion in $\lambda = g^{-2/3}$.

2.3 Coarse-graining and strong coupling expansion of the physical Hamiltonian in $\lambda = g^{-2/3}$

Introducing an UV cutoff a by considering an infinite spatial lattice of granulas $G(\mathbf{n}, a)$ at $\mathbf{x} = a\mathbf{n}$ ($\mathbf{n} \in \mathbb{Z}^3$) and averaged variables

$$S(\mathbf{n}) := \frac{1}{a^3} \int_{G(\mathbf{n}, a)} d\mathbf{x} S(\mathbf{x}) , \quad (2.9)$$

and discretised spatial derivatives, the expansion of the Hamiltonian in $\lambda = g^{-2/3}$ can be written

$$H = \frac{g^{2/3}}{a} \left[\mathcal{H}_0 + \lambda \sum_{\alpha} \mathcal{V}_{\alpha}^{(\partial)} + \lambda^2 \left(\sum_{\beta} \mathcal{V}_{\beta}^{(\Delta)} + \sum_{\gamma} \mathcal{V}_{\gamma}^{(\partial \neq \Delta)} \right) + \mathcal{O}(\lambda^3) \right]. \quad (2.10)$$

The "free" Hamiltonian $H_0 = (g^{2/3}/a) \mathcal{H}_0 + H_m = \sum_{\mathbf{n}} H_0^{QM}(\mathbf{n})$ is the sum of the Hamiltonians of Dirac-Yang-Mills quantum mechanics of constant fields in each box, and the interaction terms $\mathcal{V}^{(\partial)}, \mathcal{V}^{(\Delta)}, \dots$ lead to interactions between the granulas.

2.4 Zeroth-order: Dirac-Yang-Mills quantum mechanics of spatially constant fields

Transforming to the intrinsic system of the symmetric tensor S , with Jacobian $\sin \beta \prod_{i < j} (\phi_i - \phi_j)$,

$$S = R^T(\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R(\alpha, \beta, \gamma), \quad \psi'_{L,R} = R_{ij}^T \tilde{\psi}_{L,R}^{(j)}, \quad \psi'^{(0)}_{L,R} = \tilde{\psi}_{L,R}^{(0)}, \quad (2.11)$$

the "free" Hamiltonian in each box (volume V) takes the form [9]

$$H_0^{QM} = \frac{g^{2/3}}{V^{1/3}} \left[\mathcal{H}^G + \mathcal{H}^D + \mathcal{H}^C \right] + \frac{1}{2} m \left[\left(\tilde{\psi}_L^{(0)\dagger} \tilde{\psi}_R^{(0)} + \sum_{i=1}^3 \tilde{\psi}_L^{(i)\dagger} \tilde{\psi}_R^{(i)} \right) + h.c. \right], \quad (2.12)$$

with the glueball part \mathcal{H}^G , the minimal-coupling \mathcal{H}^D , and the Coulomb-potential-type part \mathcal{H}^C

$$\mathcal{H}^G = \frac{1}{2} \sum_{ijk}^{\text{cyclic}} \left(-\frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_i^2 - \phi_j^2} \left(\phi_i \frac{\partial}{\partial \phi_i} - \phi_j \frac{\partial}{\partial \phi_j} \right) + (\xi_i - \tilde{J}_i^Q)^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + \phi_j^2 \phi_k^2 \right), \quad (2.13)$$

$$\mathcal{H}^D = \frac{1}{2} (\phi_1 + \phi_2 + \phi_3) \left(\tilde{N}_L^{(0)} - \tilde{N}_R^{(0)} \right) + \frac{1}{2} \sum_{ijk}^{\text{cyclic}} (\phi_i - (\phi_j + \phi_k)) \left(\tilde{N}_L^{(i)} - \tilde{N}_R^{(i)} \right), \quad (2.14)$$

$$\mathcal{H}^C = \sum_{ijk}^{\text{cyclic}} \frac{\tilde{\rho}_i (\xi_i - \tilde{J}_i^Q + \tilde{\rho}_i)}{(\phi_j + \phi_k)^2}, \quad (2.15)$$

$$\text{and the total spin} \quad J_i = R_{ij}(\chi) \xi_j, \quad [J_i, H] = 0. \quad (2.16)$$

The matrix elements become

$$\langle \Phi_1 | \mathcal{O} | \Phi_2 \rangle = \int d\alpha \sin \beta d\beta d\gamma \int_{0 < \phi_1 < \phi_2 < \phi_3} d\phi_1 d\phi_2 d\phi_3 (\phi_1^2 - \phi_2^2)(\phi_2^2 - \phi_3^2)(\phi_3^2 - \phi_1^2) \int d\bar{\psi}' d\psi' \Phi_1^* \mathcal{O} \Phi_2.$$

The l.h.s. of Fig.1 shows the 0^+ energy spectrum of the lowest pure-gluon (G) and quark-gluon (QG) cases for one quark-flavor which can be calculated with high accuracy using the variational approach. The energies of the quark-gluon ground state and the sigma-antisigma excitation are lower than that of the lowest pure-gluon state. This is due to a large negative contribution from $\langle \mathcal{H}^D \rangle$, in addition to the large positive $\langle \mathcal{H}^G \rangle$, while $\langle \mathcal{H}^C \rangle \simeq 0$ (see [9] for details).

Furthermore, as a consequence of the zero-energy valleys " $\phi_1 = \phi_2 = 0$, ϕ_3 arbitrary" of the classical magnetic potential $B^2 = \phi_2^2 \phi_3^2 + \phi_3^2 \phi_1^2 + \phi_1^2 \phi_2^2$, practically all glueball excitation-energy results from an increase of expectation value of the "constant Abelian field" ϕ_3 as shown for the pure-gluon case on the r.h.s. of Fig.1 (see [7] for details).

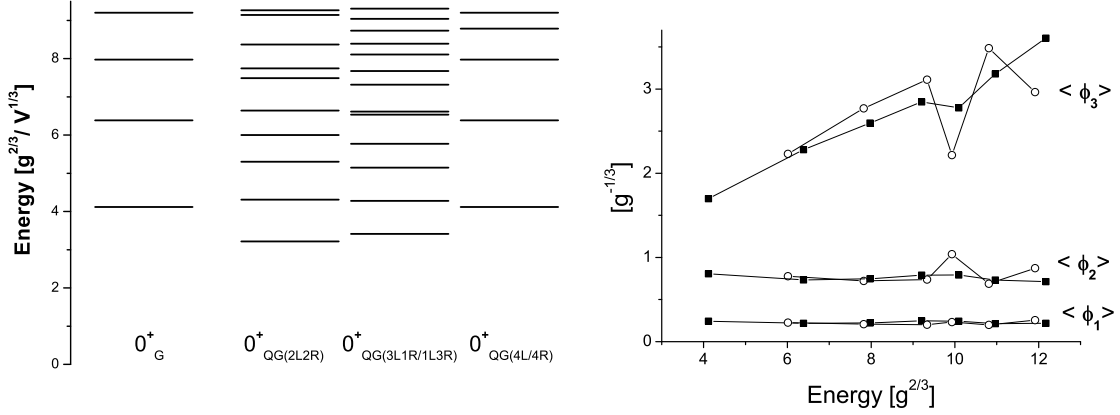


Figure 1: L.h.s.: Lowest energy levels for the pure-gluon (G) and the quark-gluon case (QG) for 2-colors and one quark flavor. The energies of the quark-gluon ground state and the sigma-antisigma excitation are lower than that of the lowest pure-gluon state. R.h.s.: (for pure-gluon case and setting $V \equiv 1$): $\langle \phi_3 \rangle$ is raising with increasing excitation, whereas $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ are practically constant, independent of whether spin-0 (dark boxes) or spin-2 states (open circles).

2.5 Perturbation theory in λ and coupling constant renormalisation in the IR

Including the interactions $\mathcal{V}^{(\partial)}$, $\mathcal{V}^{(\Delta)}$ using 1st and 2nd order perturbation theory in $\lambda = g^{-2/3}$ give the result [8] (for pure-gluon case and only including spin-0 fields in a first approximation)

$$E_{\text{vac}}^+ = \mathcal{N} \frac{g^{2/3}}{a} \left[4.1167 + 29.894\lambda^2 + \mathcal{O}(\lambda^3) \right], \quad (2.17)$$

$$E_1^{(0)+}(k) - E_{\text{vac}}^+ = \left[2.270 + 13.511\lambda^2 + \mathcal{O}(\lambda^3) \right] \frac{g^{2/3}}{a} + 0.488 \frac{a}{g^{2/3}} k^2 + \mathcal{O}((a^2 k^2)^2), \quad (2.18)$$

for the energy of the interacting glueball vacuum and the spectrum of the interacting spin-0 glueball. Lorentz invariance demands $E = \sqrt{M^2 + k^2} \simeq M + \frac{1}{2M} k^2$, which is violated in this 1st approximation by a factor of 2. In order to get a Lorentz invariant result, $J = L + S$ states should be considered including also spin-2 states and the general $\mathcal{V}^{(\partial\partial)}$.

Independence of the physical glueball mass

$$M = \frac{g_0^{2/3}}{a} \left[\mu + c g_0^{-4/3} \right] \quad (2.19)$$

of box size a , one obtains

$$\gamma(g_0) \equiv a \frac{d}{da} g_0(a) = \frac{3}{2} g_0 \frac{\mu + c g_0^{-4/3}}{\mu - c g_0^{-4/3}} \quad (2.20)$$

which vanishes for $g_0 = 0$ (pert. fixed point) or $g_0^{4/3} = -c/\mu$ (IR fixed point, if $c < 0$). For $c > 0$

$$\text{for } c > 0: \quad g_0^{2/3}(Ma) = \frac{Ma}{2\mu} + \sqrt{\left(\frac{Ma}{2\mu} \right)^2 - \frac{c}{\mu}}, \quad a > a_c := 2\sqrt{c\mu}/M \quad (2.21)$$

My (incomplete) result $c_1^{(0)}/\mu_1^{(0)} = 5.95$ suggests, that no IR fixed points exist. critical coupling $g_0^2|_c = 14.52$ and $a_c \sim 1.4$ fm for $M \sim 1.6$ GeV.

3. Symmetric gauge for SU(3)

Using the idea of *minimal embedding* of $su(2)$ in $su(3)$ by Kihlberg and Marnelius [4]

$$\begin{aligned}
 \tau_1 := \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \tau_2 := -\lambda_5 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} & \tau_3 := \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 \tau_4 := \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \tau_5 := \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \tau_6 := \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 \tau_7 := \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \tau_8 := \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
 \end{aligned} \tag{3.1}$$

such that the corresponding non-trivial non-vanishing structure constants, $[\frac{\tau_a}{2}, \frac{\tau_b}{2}] = ic_{abc} \frac{\tau_c}{2}$, have at least one index $\in \{1, 2, 3\}$, the symmetric gauge, Equ.(2.2), can be generalised to $SU(3)$ [5, 10],

$$\chi_a(A) = \sum_{b=1}^8 \sum_{i=1}^3 c_{abi} A_{bi} = 0, \quad a = 1, \dots, 8 \quad (\text{"symmetric gauge" for } SU(3)). \tag{3.2}$$

Carrying out the coordinate transformation [10]

$$A_{ak}(q_1, \dots, q_8, \hat{S}) = O_{a\hat{a}}(q) \hat{S}_{\hat{a}k} - \frac{1}{2g} c_{abc} (O(q) \partial_k O^T(q))_{bc}, \quad \psi_\alpha(q_1, \dots, q_8, \psi^{RS}) = U_{\alpha\hat{\beta}}(q) \psi_{\hat{\beta}}^{RS}$$

$$\hat{S}_{\hat{a}k} \equiv \begin{pmatrix} S_{ik} \\ \bar{S}_{Ak} \end{pmatrix} = \begin{pmatrix} S_{ik} \text{ pos. def.} \\ \begin{array}{ccc} W_0 & X_3 - W_3 & X_2 + W_2 \\ X_3 + W_3 & W_0 & X_1 - W_1 \\ X_2 - W_2 & X_1 + W_1 & W_0 \\ -\frac{\sqrt{3}}{2} Y_1 - \frac{1}{2} W_1 & \frac{\sqrt{3}}{2} Y_2 - \frac{1}{2} W_2 & W_3 \\ -\frac{\sqrt{3}}{2} W_1 - \frac{1}{2} Y_1 & \frac{\sqrt{3}}{2} W_2 - \frac{1}{2} Y_2 & Y_3 \end{array} \end{pmatrix}, \quad c_{\hat{a}\hat{b}\hat{c}} \hat{S}_{\hat{b}k} = 0, \tag{3.3}$$

an unconstrained Hamiltonian formulation of QCD can be obtained. The existence and uniqueness of (3.3) can be investigated by solving the 16 equs.

$$\hat{S}_{\hat{a}i} \hat{S}_{\hat{a}j} = A_{ai} A_{aj} \text{ (6 equs.)} \quad \wedge \quad d_{\hat{a}\hat{b}\hat{c}} \hat{S}_{\hat{a}i} \hat{S}_{\hat{b}j} \hat{S}_{\hat{c}k} = d_{abc} A_{ai} A_{bj} A_{ck} \text{ (10 equs.)} \tag{3.4}$$

for the 16 components of \hat{S} in terms of 24 given components A .

Analysing the Gauss law operators and the unconstrained angular momentum operators in terms of the new variables in analogy to the 2-color case, it can be shown that the original constrained 24 colored spin-1 gluon fields A and the 12 colored spin-1/2 quark fields ψ (per flavor) reduce to 16 physical colorless spin-0, spin-1, spin-2, and spin-3 glueball fields (the 16 components of \hat{S}) and a colorless spin-3/2 Rarita-Schwinger field ψ^{RS} (12 components per flavor). As for the 2-color case, the gauge reduction converts *color* \rightarrow *spin*, which might have important consequences for low energy Spin-Physics. In terms of the colorless Rarita-Schwinger fields the $\Delta^{++}(3/2)$ could have the spin content $(+3/2, +1/2, -1/2)$ in accordance with the Spin-Statistics-Theorem.

Transforming to the intrinsic system of the embedded upper part S of \widehat{S} (see [10] for details)

$$S = R^T(\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R(\alpha, \beta, \gamma), \quad \wedge \quad X_i \rightarrow x_i, \quad Y_i \rightarrow y_i, \dots, \quad \wedge \quad \psi^{RS} \rightarrow \widetilde{\psi}^{RS}, \quad (3.5)$$

one finds that the magnetic potential B^2 has the zero-energy valleys ("constant Abelian fields")

$$B^2 = 0 \quad : \quad \phi_3 \text{ and } y_3 \text{ arbitrary} \quad \wedge \quad \text{all others zero} \quad (3.6)$$

Hence, practically all glueball excitation-energy should result from an increase of expectation values of these two "constant Abelian fields", in analogy to $SU(2)$. Furthermore, at the bottom of the valleys the important minimal-coupling-interaction of $\widetilde{\psi}^{RS}$ (analogous to (2.14)) becomes diagonal

$$\mathcal{H}_{\text{diag}}^D = \frac{1}{2} \widetilde{\psi}_L^{(1, \frac{1}{2})\dagger} [(\phi_3 \lambda_3 + y_3 \lambda_8) \otimes \sigma_3] \widetilde{\psi}_L^{(1, \frac{1}{2})} - \frac{1}{2} \widetilde{\psi}_R^{(\frac{1}{2}, 1)\dagger} [\sigma_3 \otimes (\phi_3 \lambda_3 + y_3 \lambda_8)] \widetilde{\psi}_R^{(\frac{1}{2}, 1)}. \quad (3.7)$$

Due to the difficulty of the FP-determinant (see [10]), precise calculations are not possible yet. Note, however, that in one spatial dimension the symmetric gauge for $SU(3)$ reduces to

$$A^{(1d)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \\ 0 & 0 & A_{43} \\ 0 & 0 & A_{53} \\ 0 & 0 & A_{63} \\ 0 & 0 & A_{73} \\ 0 & 0 & A_{83} \end{pmatrix} \rightarrow \widehat{S}^{(1d)} = \widehat{S}_{\text{intrinsic}}^{(1d)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \phi_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \quad (3.8)$$

which consistently reduces the Equ.(3.4) for given A_3 to

$$\phi_3^2 + y_3^2 = A_{a3} A_{a3} \quad \wedge \quad \phi_3^2 y_3 - 3 y_3^3 = d_{abc} A_{a3} A_{b3} A_{c3} \quad (3.9)$$

with 6 solutions separated by zero-lines of the FP-determinant ("Gribov-horizons"). Exactly one solution exists in the "fundamental domain" $0 < \phi_3 < \infty \wedge \phi_3/\sqrt{3} < y_3 < \infty$, and we can replace

$$\int_{-\infty}^{+\infty} \prod_{a=1}^8 dA_{a3} \rightarrow \int_0^\infty d\phi_3 \int_{\phi_3/\sqrt{3}}^\infty dy_3 \phi_3^2 (\phi_3^2 - 3y_3^2)^2 \propto \int_0^\infty r dr \int_{\pi/6}^{\pi/2} d\psi \cos^2(3\psi). \quad (3.10)$$

For two spatial dimensions, one can show that (putting in Equ.(3.3) $W_1 \equiv X_1, W_2 \equiv -X_2$)

$$A^{(2d)} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & 0 \\ A_{41} & A_{42} & 0 \\ A_{51} & A_{52} & 0 \\ A_{61} & A_{62} & 0 \\ A_{71} & A_{72} & 0 \\ A_{81} & A_{82} & 0 \end{pmatrix} \rightarrow \widehat{S}_{\text{intrinsic}}^{(2d)} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & 0 \\ \hline 0 & x_3 & 0 \\ x_3 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ -\frac{\sqrt{3}}{2}y_1 - \frac{1}{2}x_1 & \frac{\sqrt{3}}{2}y_2 + \frac{1}{2}x_2 & 0 \\ -\frac{\sqrt{3}}{2}y_1 + \frac{1}{2}x_1 & -\frac{\sqrt{3}}{2}y_2 + \frac{1}{2}x_2 & 0 \end{pmatrix} \quad (3.11)$$

consistently reduces (3.4) to a system of 7 eqs. for 8 physical fields (incl. rot.-angle γ), which, adding as an 8th eq. $(d_{\hat{a}\hat{b}\hat{c}} \widehat{S}_{\hat{b}1} \widehat{S}_{\hat{c}2})^2 = (d_{abc} A_{b1} A_{c2})^2$, can be solved numerically for randomly generated $A^{(2d)}$, again yielding solutions separated by horizons. Restricting to a fundamental domain

$$\int_{-\infty}^{+\infty} \prod_{a,b=1}^8 dA_{a1} dA_{b2} \rightarrow \int d\gamma \int_{0 < \phi_1 < \phi_2 < \infty} d\phi_1 d\phi_2 (\phi_1 - \phi_2) \int_{R_1(\phi_1, \phi_2)} dx_1 dx_2 dx_3 \int_{R_2(x_1, x_2, x_3, \phi_1, \phi_2)} dy_1 dy_2 \mathcal{J}$$

Due to the difficulty of the FP-determinant, I have, however, not yet succeeded in a satisfactory description of the regions R_1 and R_2 . For the general case of three dimensions, I have found several solutions of the Equ.(3.4) numerically for a randomly generated A , but to write the corresponding unconstrained integral over a fundamental domain is a difficult, but I think solvable, future task.

4. Conclusions

Using a canonical transformation of the dynamical variables, which Abelianises the non-Abelian Gauss-law constraints to be implemented, a reformulation of QCD in terms of gauge invariant dynamical variables can be achieved. The exact implementation of the Gauss laws reduces the colored spin-1 gluons and spin-1/2 quarks to unconstrained colorless spin-0, spin-1, spin-2 and spin-3 glueball fields and colorless Rarita-Schwinger fields respectively. The obtained physical Hamiltonian admits a systematic strong-coupling expansion in powers of $\lambda = g^{-2/3}$, equivalent to an expansion in the number of spatial derivatives. The leading-order term in this expansion corresponds to non-interacting hybrid-glueballs, whose low-lying masses can be calculated with high accuracy (at the moment only for the unphysical, but technically much simpler 2-color case) by solving the Schrödinger-equation of Dirac-Yang-Mills quantum mechanics of spatially constant fields. Higher-order terms in λ lead to interactions between the hybrid-glueballs and can be taken into account systematically, using perturbation theory in λ , allowing for the study of the difficult questions of Lorentz invariance and coupling constant renormalisation in the IR.

References

- [1] N.H. Christ and T.D. Lee, *Operator ordering and Feynman rules in gauge theories*, *Phys. Rev. D* **22** (1980) 939.
- [2] M. Lüscher and G. Münster, *Weak coupling expansion of the low-lying energy values in the SU(2) gauge theory on a torus*, *Nucl. Phys. B* **232** (1984) 445.
- [3] P. Weisz and V. Ziemann, *Weak coupling expansion of the low-lying energy values in SU(3) gauge theory on a torus*, *Nucl. Phys. B* **284** (1987) 157.
- [4] A. Kihlberg and R. Marnelius, *Properties of Yang-Mills theories with gauge-fixing conditions on the field strength*, *Phys. Rev. D* **26** (1982) 2003.
- [5] B. Dahmen and B. Raabe, *Unconstrained SU(2) and SU(3) Yang-Mills classical mechanics*, *Nucl. Phys. B* **384** (1992) 352.
- [6] A.M. Khvedelidze and H.-P. Pavel, *Unconstrained Hamiltonian formulation of SU(2) gluodynamics*, *Phys. Rev. D* **59** (1999) 105017 [hep-th/9808102].
- [7] H.-P. Pavel, *SU(2) Yang-Mills quantum mechanics of spatially constant fields*, *Phys. Lett. B* **648** (2007) 97 [hep-th/0701283].
- [8] H.-P. Pavel, *Expansion of the Yang-Mills Hamiltonian in spatial derivatives and glueball spectrum*, *Phys. Lett. B* **685** (2010) 353 [arXiv:0912.5465[hep-th]].
- [9] H.-P. Pavel, *SU(2) Dirac-Yang-Mills quantum mechanics of spatially constant quark and gluon fields*, *Phys. Lett. B* **700** (2011) 265 [arXiv:1104.1576[hep-th]].
- [10] H.-P. Pavel, *Unconstrained Hamiltonian formulation of low energy SU(3) Yang-Mills quantum theory*, arXiv: 1205.2237 [hep-th] (2012).